South East Asian J. of Mathematics and Mathematical Sciences
Vol. 22, No. 1 (2026), pp. 225-248

DOI: 10.56827/SEAJMMS.2026.2201.16 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

GEOMETRICAL PROPERTIES OF GENERALIZED QUASI-CONFORMAL CURVATURE TENSOR

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(**Received:** *May 22, 2025* **Accepted:** *Mar. 18, 2026* **Published:** *Apr. 30, 2026*)

Abstract: This paper examines the generalized quasi-conformal curvature $\mathcal{G}$, which generalizes the concept of conformal curvature tensor $\mathcal{C}$, quasi-conformal curvature tensor $\widetilde{\mathcal{C}}$, projective curvature tensor $\mathcal{P}$, pseudo projective curvature tensor $\widetilde{\mathcal{P}}$, and pseudo W_2-curvature tensor $\widetilde{\mathcal{W}}_2$. Initially, we acquire some geometrical features. Subsequently, we examine pseudo generalized conformal symmetric manifolds. Divergence-free generalized quasi-conformal curvature tensor is derived from the Gray's decomposition. Additionally, we also examine Einstein $(P\mathcal{G}S)_n$ manifolds. A study of generalized quasi-conformal has been conducted as the four-dimensional spacetime of general relativity $\widetilde{GR}$. Ultimately, we examine a non-trivial Lorentzian metric of $(P\mathcal{G}S)_4$.

Keywords and Phrases: Pseudo symmetric, perfect fluid, Einstein's field equation, Quasi-conformal curvature tensor $\widetilde{\mathcal{C}}$, pseudo projective curvature tensor $\widetilde{\mathcal{P}}$, conformal curvature tensor $\mathcal{C}$, projective curvature tensor $\mathcal{P}$, concircular curvature tensor $\mathcal{L}$, generalized quasi conformal curvature tensor $\mathcal{G}$, Gray's decomposition.

2020 Mathematics Subject Classification: 53C25, 53D15.

1. Introduction

A non-flat pseudo Riemannian manifold is said to be pseudo symmetric if its curvature R are of the type (0,4) satisfies the condition:

$$\begin{aligned}(D_X R)(Y,Z,U,V) =& 2A(X)R(Y,Z,U,V) + A(Y)R(X,Z,U,V) + \\ & A(Z)R(Y,X,U,V) + A(U)R(Y,Z,X,V) \\ & + A(V)R(Y,Z,U,X),\end{aligned}$$

$\forall\ X,Y,Z,U,V \in$TM, where A is a non-zero 1-form, ρ is a vector field defined by $g(X,\rho) = A(X)$ for all X, D is the operator of covariant differentiation with respect to the metric g where $R(Y,Z,U,V) = g(\mathbb{R}(\mathbb{Y},\mathbb{Z})\mathbb{U},V), \mathbb{R}$ is the curvature tensor of the type (1,3). The 1-form A is called the associated 1-form of the manifold. If $A = 0$, then, the manifold reduces to a locally symmetric manifold in the sense of Cartan. An n-dimensional pseudo symmetric manifold is denoted by $(PS)_n$. Pseudo symmetric manifolds have been investigated by several authors.

Lorentzian manifold is a special category of semi-Riemannian manifold equipped with a Lorentzian metric g. The spacetime of $\widetilde{GR}$ (general relativity) is nothing but a time-oriented connected Lorentzian manifold M^n with the signature $(+,+,+,-)$. The study of the causal character of vectors of the Lorentzian manifold begins with the geometry of the Lorentzian metric. The investigation of general relativity provides a natural framework for this causality of the Lorentzian manifold.

A perfect fluid (PF), which performs a significant role in $\widetilde{GR}$, does not have heat conduction terms or the stress terms corresponding to viscosity [12]. Hence, for a PF, the energy momentum tensor (EMT) $\mathcal{T}$ is of the type (0,2) given by the expression:

$$\mathcal{T}(X,Y) = pg(X,Y) + (\sigma + p)A(X)A(Y), \tag{1.1}$$

in which σ and p stand for the energy density and the isotropic pressure [22] and ρ is a unit time like vector field ($g(\rho,\rho) = -1$) metrically equivalent to the 1-form A.

The Einstein's field equation (EFE) without cosmological constant is written by

$$Ric(X,Y) - \frac{r}{2}g(X,Y) = k\mathcal{T}(X,Y), \tag{1.2}$$

where r and k are the scalar curvature and gravitational constant, respectively. Montica and Suh [16] investigated pseudo $\widetilde{Z}$-symmetric spacetimes, while Ozen [23] studied M-projectively flat spacetimes. A condition under which a pseudo-symmetric spacetime becomes a PF-spacetime was recently established by Zhao et al. [33]. Furthermore, pseudo semi-projective symmetric manifolds were studied by De and Majhi [6]. The concept of ψ-conharmonically symmetric spacetime was introduced by De and De [7], and ψ-conformally symmetric spacetime was investigated by Mofarreh, De, and De [20]. In addition, several authors have studied spacetimes of $\widetilde{GR}$ in various contexts; for more details, see [9, 17], etc.

The Weyl conformal tensor is a well-known tensor, which is invariant under every conformal transformation [8]. In particular, if a conformal transformation transforms a harmonic function, then it is called a conharmonic transformation, and the conharmonic curvature tensor [14] is invariant under such transformations. It is well known that a semi-Riemannian manifold is locally projectively flat if and only if the projective curvature tensor vanishes. Apart from the conformal and projective curvature tensors, the concircular curvature tensor also plays an important role in the semi-Riemannian setting. Later, Yano and Sawaki [32] generalized the conformal curvature tensor $\mathcal{C}$ and the concircular curvature tensor $\mathcal{L}$ to the quasi-conformal curvature tensor $\widetilde{\mathcal{C}}$, whereas Prasad [24] introduced the pseudo-projective curvature tensor $\widetilde{\mathcal{P}}$ on a Riemannian manifold as a combination of the projective curvature tensor $\mathcal{P}$ and the concircular curvature tensor $\mathcal{L}$, both of type (0,4), as follows:

$$\widetilde{\mathcal{C}}(X,Y,Z,W) = -(n-2)b\mathcal{C}(X,Y,Z,W) + [a+(n-2)b]\mathcal{L}(X,Y,Z,W),$$

$$\widetilde{\mathcal{P}}(X,Y,Z,W) = -(n-1)\mathcal{P}(X,Y,Z,W) + [a+(n-1)b]\mathcal{L}(X,Y,Z,W),$$

where

$$\begin{aligned} &\widetilde{\mathcal{P}}(X,Y,Z,W) = g(\widetilde{\mathbb{P}}(X,Y)Z,W), \widetilde{\mathcal{C}}(X,Y,Z,W) = g(\widetilde{\mathbb{C}}(X,Y)Z,W),\\ &\mathcal{L}(X,Y,Z,W) = g(\mathbb{L}(X,Y)Z,W), \mathcal{P}(X,Y,Z,W) = g(\mathbb{P}(X,Y)Z,W)\\ \textit{and}\ \ &\mathcal{C}(X,Y,Z,W) = g(\mathbb{C}(X,Y)Z,W). \end{aligned}$$

Quasi-conformal curvature tensor and pseudo-projective curvature tensor on different spaces have been studied in several ways by numerous writers as seen in [26], [13], [21] etc.

In 2011, Prasad, Doulo and Pandey [26] extended the concept of quasi conformal curvature tensor $\widetilde{\mathbb{C}}$ of the type (1,3) as follows which contains eight known curvature tensors;

$$\begin{aligned} \mathbb{G}(X,Y)U =& a\mathbb{R}(X,Y)U + b[Ric(Y,U)X - Ric(X,U)Y] +\\ & c[g(Y,U)QX - g(X,U)QY] -\\ & \frac{r}{n}\left(\frac{a}{n-1} + b + c\right)[g(Y,U)X - g(X,U)Y], \end{aligned} \tag{1.3}$$

where a, b, and c are non-zero real constants on (M^n, g); Ric, Q, and r denote the Ricci tensor of type $(0, 2)$, the Ricci operator of type $(1, 1)$, and the scalar curvature, respectively. Such a tensor $\mathbb{G}(X, Y)U$ is known as "Generalized quasi-conformal curvature tensor".
In particular, the tensor $\mathbb{G}$ reduces to:

(i) the curvature tensor $\mathbb{R}$ if $a = b = c = 0$,

(ii) the conformal curvature tensor $\mathbb{C}$ [19] if $a = 1, b = c = -\frac{1}{n-2}$,

(iii) the quasi-conformal curvature tensor $\widetilde{\mathbb{C}}$ [32] if $b = c$,

(iv) the projective curvature tensor $\mathbb{P}$ [19] if $a = 1, b = -\frac{1}{n-1}, c = 0$,

(v) the concircular curvature tensor $\mathbb{L}$ [31] if $a = 1, b = c = 0$,

(vi) the W_2-curvature tensor $\mathbb{W}_2$ [18] if $a = 1, b = 0, c = -\frac{1}{n-1}$,

(vii) the pseudo projective curvature tensor $\widetilde{\mathbb{P}}$ [24] if $c = 0$,

(viii) the pseudo W_2-curvature tensor $\widetilde{\mathbb{W}}_2$ [25] if $b = 0$.

Equation (1.3) can be put as

$$\begin{aligned}\mathcal{G}(X, Y, U, V) =& aR(X, Y, U, V) + b[Ric(Y, U)g(X, V)- \\ & Ric(X, U)g(Y, V)] + c[g(Y, U)Ric(X, V)- \\ & g(X, U)Ric(Y, V)] - \frac{r}{n}\left(\frac{a}{n-1} + b + c\right) \times \\ & [g(Y, U)g(X, V) - g(X, U)g(Y, V)],\end{aligned} \tag{1.4}$$

where $\mathcal{G}(X, Y, U, V) = g(\mathbb{G}(X, Y)U, V)$.

The generalized quasi-conformal curvature tensor on K-contact manifolds has been investigated by Gupta and Prasad in 2020 [11], as well as by many other authors.

A non-flat semi-Riemannian manifold (M^n, g), with $n > 2$, is said to be pseudo generalized quasi-conformally symmetric if the tensor $\mathcal{G}$ of type $(0, 4)$ satisfies the following condition:

$$\begin{aligned}(D_X\mathcal{G})(Y, Z, U, V) =& 2A(X)\mathcal{G}(Y, Z, U, V) + A(Y)\mathcal{G}(X, Z, U, V)+ \\ & A(Z)\mathcal{G}(Y, X, U, V) + A(U)\mathcal{G}(Y, Z, X, V) \\ & + A(V)\mathcal{G}(Y, Z, U, X),\end{aligned} \tag{1.5}$$

where A is a non-zero 1-form and ρ is a vector field defined by $g(X,\rho) = A(X)$.

An n-dimensional pseudo generalized quasi-conformally symmetric manifold is denoted by $(P\mathcal{G}S)_n$, where P stands for pseudo, $\mathcal{G}$ stands for the generalized quasi-conformal curvature tensor and S indicates symmetric.

In particular, we have

(i) if $a = b = c = 0$, then $(P\mathcal{G}S)_n \Longrightarrow (PS)_n$ i.e. pseudo symmetric [1],

(ii) if $a = 1, b = c = -\frac{1}{n-2}$, then $(P\mathcal{G}S)_n \Longrightarrow (P\mathcal{C}S)_n$ i.e. pseudo conformally symmetric [4],

(iii) if $a = 1, b = -\frac{1}{n-1}, c = 0$, then $(P\mathcal{G}S)_n \Longrightarrow (P\mathcal{P}S)_n$ i.e. pseudo projective symmetric [2],

(iv) if $a = 1, b = c = 0$, then $(P\mathcal{G}S)_n \Longrightarrow (P\mathcal{L}S)_n$ i.e. pseudo concircular symmetric ([3] and [23])

(v) if $a = 1, b = 0, c = -\frac{1}{n-1}$, then $(P\mathcal{G}S)_n \Longrightarrow (P\mathcal{W}_2S)_n$ i.e. pseudo W_2 symmetric [5].

The present paper is organized as: After introduction in section 2, we study some basic geometric properties of generalized quasi-conformal curvature tensor $\mathcal{G}$. A classification theorem for a Riemannian manifold admitting a divergence-free generalized quasi-conformal curvature tensor $\mathcal{G}$ given in each subspace of Gray's decomposition of the covariant derivative of the Ricci tensor in section 3. Some properties of $(P\mathcal{G}S)_n$ manifold are given in section 4 and 5, respectively. In section 6, we analyze Einstein $(P\mathcal{G}S)_n$. Section 7 is devoted to study $(P\mathcal{G}S)_4$ spacetimes. Finally, we construct a non-trivial example of $(P\mathcal{G}S)_4$.

2. Elementary Properties of generalized quasi-conformal curvature tensor $\mathcal{G}$

From (1.3), $\mathbb{G}$ fulfills the following algebraic properties:

$$\mathbb{G}(X,Y)U + \mathbb{G}(Y,X)U = 0, \tag{2.1}$$

and

$$\mathbb{G}(X,Y)U + \mathbb{G}(Y,U)X + \mathbb{G}(U,X)Y = 0. \tag{2.2}$$

It is also clear from (1.4), that

$$\sum_{i=1}^{n} \mathcal{G}(X,Y,e_i,e_i) = \sum_{i=1}^{n} \mathcal{G}(e_i,e_i,U,V) = 0, \tag{2.3}$$

$$\begin{aligned}\sum_{i=1}^{n}\mathcal{G}(e_i,Y,U,e_i) &= [a+(n-1)b-c]\left[Ric(Y,U)-\frac{r}{n}g(Y,U)\right]\\ &= \mathcal{G}_1(Y,U),\end{aligned}\tag{2.4}$$

and

$$\begin{aligned}\sum_{i=1}^{n}\mathcal{G}(X,e_i,e_i,V) &= [a-b+(n-1)c]\left[Ric(X,V)-\frac{r}{n}g(X,V)\right]\\ &= \mathcal{G}_2(X,V),\end{aligned}\tag{2.5}$$

where at each point of the manifold $\{e_i, i=1,2,...,n\}$ be an orthonormal basis of the tangent space, $r=\sum_{i=1}^{n} Ric(e_i,e_i)$.

From (1.4) and (2.1), we get

$$\text{(i)}\ \ \mathcal{G}(X,Y,U,V)+\mathcal{G}(Y,X,U,V)=0,\tag{2.6}$$

$$\text{(ii)}\ \ \mathcal{G}(X,Y,U,V)+\mathcal{G}(X,Y,V,U)\neq 0,$$

$$\text{(iii)}\ \ \mathcal{G}(X,Y,U,V)-\mathcal{G}(U,V,X,Y)\neq 0,$$

$$\text{(iv)}\ \ \mathcal{G}(X,Y,U,V)+\mathcal{G}(Y,U,X,V)+\mathcal{G}(U,X,Y,V)=0.\tag{2.7}$$

Proposition 2.1. *If $\mathcal{G}$ curvature tensor is symmetric in the sense of Cartan, the scalar curvature is constant, provided $[an+2(n-1)(b+c)]\neq 0$.*
Proof. Differentiating covariantly (1.3) with respect to X, we get

$$\begin{aligned}(D_X\mathbb{G})(Y,Z)U =& a(D_X\mathbb{R})(Y,Z)U+b[(D_XRic)(Z,U)Y-\\ &(D_XRic)(Y,U)Z]+c[g(Z,U)(D_XQ)(Y)-\\ &g(Y,U)(D_XQ)(Z)]-\frac{dr(X)}{n}\left(\frac{a}{n-1}+b+c\right)\times\\ &[g(Z,U)Y-g(Y,U)Z].\end{aligned}\tag{2.8}$$

According to our assumption, we have from (2.8)

$$\begin{aligned}-a(D_X\mathbb{R})(Y,Z)U =& b[(D_XRic)(Z,U)Y-(D_XRic)(Y,U)Z]+\\ &c[g(Z,U)(D_XQ)(Y)-g(Y,U)(D_XQ)(Z)]-\\ &\frac{dr(X)}{n}\left(\frac{a}{n-1}+b+c\right)[g(Z,U)Y-g(Y,U)Z].\end{aligned}\tag{2.9}$$

Contracting (2.9) with respect to Y, we get

$$\begin{aligned}-a(D_X Ric)(Z,U) =&[(n-1)b-c](D_X Ric)(Z,U)+cg(Z,U)(D_X r)\\ &-\frac{dr(X)}{n}\left(\frac{a}{n-1}+b+c\right)(n-1)g(Z,U).\end{aligned} \tag{2.10}$$

Again, contracting (2.10) with respect to Z and U, we have

$$dr(X)=0, \quad \textit{provided} \quad [an+2(n-1)(b+c)]\neq 0.$$

This proves Proposition 2.1.

3. The divergence of generalized quasi-conformal curvature tensor $\mathbb{G}$ under Gray's decomposition

Considering the action of the orthogonal group on the space of tensors with the symmetries of the covariant derivative of the Ricci curvature, Gray decomposed this space into irreducible components [10]. Gray proposed that the covariant derivative of the Ricci tensor, that is, $D\cdot Ric$, can be decomposed into $O(X)$-invariant terms. According to him, the covariant derivative of the Ricci tensor can be converted into $O(X)$-invariant terms as follows [17].

$$\begin{aligned}(D_X Ric)(Y,Z) =&\widehat{R}(X,Y)Z+\frac{n(D_X r)}{(n-1)(n+2)}g(Y,Z)\\ &+\frac{(n-2)(D_Y r)}{2(n-1)(n+2)}g(X,Z)+\frac{(n-2)(D_Z r)}{2(n-1)(n+2)}g(X,Y).\end{aligned} \tag{3.1}$$

for all vector fields X,Y,Z and $\widehat{R}(X,Y)Z=\widehat{R}(X,Z)Y$ is a tensor with zero trace that can be written as a sum of its orthogonal components:

$$\begin{aligned}\widehat{R}(X,Y)Z =&\frac{1}{3}[\widehat{R}(X,Y)Z+\widehat{R}(Y,Z)X+\widehat{R}(Z,X)Y]\\ &+\frac{1}{3}[\widehat{R}(X,Y)Z-\widehat{R}(Y,X)Z]\\ &+\frac{1}{3}[\widehat{R}(X,Y)Z-\widehat{R}(Z,X)Y].\end{aligned} \tag{3.2}$$

The decompositions (3.1) and (3.2) yield $O(X)-$invariant subspace, which is characterized by linear invariant equations in $(D_X Ric)(Y,Z)$.

Therefore, the relation between $(D_X Ric)(Y,Z)$ and the divergence of conformal curvature tensor $\mathcal{C}$ can be given by the equation

$$(div\mathcal{C})(X,Y)Z=\frac{n-3}{n-2}[R(X,Y)Z-R(Y,X)Z], \text{ where } n>3. \tag{3.3}$$

The subspace in Gray's decomposition are as follows:

i. The trivial subspace is given by $(D_X Ric)(Y, Z) = 0$.

ii. The subspace $\mathcal{I}$ is characterized by $R(X, Y)Z = 0$, i.e,

$$\begin{aligned}(D_X Ric)(Y, Z) =& \frac{n(D_X r)}{(n-1)(n+2)} g(Y, Z) + \frac{(n-2)(D_Y r)}{2(n-1)(n+2)} g(X, Z) \\ &+ \frac{(n-2)(D_Z r)}{2(n-1)(n+2)} g(X, Y).\end{aligned} \tag{3.4}$$

Manifolds satisfying equation (3.4) are called Sinyukov manifolds [28].

iii. The orthogonal complements $\mathcal{I}'$, which is also called the subspace $\mathcal{A}$, is defined by

$$(D_X Ric)(Y, Z) + (D_Y Ric)(Z, X) + (D_Z Ric)(X, Y) = 0, \tag{3.5}$$

which yields that the scalar curvature r is constant. Also, the Ricci tensor is Killing tensor [30] if equation (3.5) is satisfied.

iv. In the subspaces $\mathcal{B}$ and $\mathcal{B}'$, the Ricci tensor is of Codazzi type, i.e,

$$(D_X Ric)(Y, Z) - (D_Y Ric)(X, Z) = 0. \tag{3.6}$$

v. In the subspace $\mathcal{I} \oplus \mathcal{A}$, the Ricci tensor satisfies the following cyclic condition:

$$\begin{aligned}&(D_X Ric)(Y, Z) + (D_Y Ric)(Z, X) + (D_Z Ric)(X, Y) = \\ &2\frac{dr(X)}{n+2} g(Y, Z) + 2\frac{dr(Y)}{n+2} g(X, Z) + 2\frac{dr(Z)}{n+2} g(X, Y),\end{aligned} \tag{3.7}$$

that is, the Ricci tensor is conformal Killing [27].

vi. The Ricci tensor fulfills the following Codazzi condition in the subspace $\mathcal{I} \oplus \mathcal{B}$,

$$\begin{aligned}(D_Y Ric)(Z, U) - (D_Z Ric)(Y, U) =& \frac{1}{2(n-1)} [g(Z, U) dr(Y) \\ &- g(Y, U) dr(Z)],\end{aligned} \tag{3.8}$$

which given $div\ \mathcal{C} = 0$.

vii. In the subspace $\mathcal{A} \oplus \mathcal{B}$, the scalar curvature is covariant constant.

Let us consider each of these seven cases separately.

Case (i): Contracting (2.8) with respect to X, we get

$$\begin{aligned}(div\mathbb{G})(Y,Z)U =& a(div\mathbb{R})(Y,Z)U + b[(D_Y Ric)(Z,U)- \\ & (D_Z Ric)(Y,U)] + \left[\frac{c}{2} + \frac{1}{n}\left(\frac{a}{n-1} + b + c\right)\right] \times \\ & [g(Z,U)dr(Y) - g(Y,U)dr(Z)].\end{aligned} \tag{3.9}$$

With condition **(i)**, i.e., $(D_X Ric)(Y,Z) = 0$, and using (3.9), we obtain
The generalized quasi-conformal curvature tensor $\mathbb{G}$ is divergence-free. But the converse is not generally true.

Case (ii): A semi-Riemannian manifold (M^n, g) is in subspace $\mathcal{I}$. Then in view of (3.4) and (3.9), we get

$$\begin{aligned}(div\mathbb{G})(Y,Z)U =& \Big[\frac{a+b}{2(n-1)} + \frac{c}{2} - \frac{1}{n}\left(\frac{a}{n-1} + b + c\right)\Big][g(Z,U)dr(Y) \\ & - g(Y,U)dr(Z)].\end{aligned} \tag{3.10}$$

Assuming that $(div\mathbb{G})(Y,Z)U = 0$, suitable contractions of (3.10) with respect to Z and U gives the following expression::

$$\Big[\frac{a+b}{2(n-1)} + \frac{c}{2} - \frac{1}{n}\left(\frac{a}{n-1} + b + c\right)\Big](n-1)dr(Y) = 0.$$

Hence, we have the following theorem:

Theorem 3.1. *If (M^n, g) be a semi-Riemannian manifold whose Ricci tensor Ric satisfies equation (3.4). Then, the curvature tensor $\mathbb{G}$ is divergence-free if and only if either scalar curvature r is constant or*

$$\Big[\frac{a+b}{2(n-1)} + \frac{c}{2} - \frac{1}{n}\left(\frac{a}{n-1} + b + c\right)\Big] = 0.$$

Case (iii): A semi-Riemannian manifold (M^n, g) is in subspace $\mathcal{A}$ if the Ricci tensor is Killing i.e. $(D_X Ric)(Y,Z) + (D_Y Ric)(Z,X) + (D_Z Ric)(X,Y) = 0$. This scalar curvature is constant, i.e. $dr(X) = 0$. Using this properties in (3.9), we get

$$(div\mathbb{G})(Y,Z)U = (a+b)[(D_Y Ric)(Z,U) - (D_Z Ric)(Y,U)]. \tag{3.11}$$

Conversely, if (3.11) exist, then from (3.9), we get

$$dr(X) = 0.$$

Hence, we have the following theorem:

Theorem 3.2. *If* (M^n, g) *be a semi-Riemannian manifold whose Ricci tensor is Killing. Then* $(div\mathbb{G})(Y,Z)U = (a+b)[(D_Y Ric)(Z,U) - (D_Z Ric)(Y,U)]$ *if and only if scalar curvature is constant.*

Case (iv): The next subspace $\mathcal{B}$ and $\mathcal{B}'$ contains manifold whose Ricci tensor is Codazzi type i.e. $(D_X Ric)(Y,Z) = (D_Y Ric)(X,Z)$. Then, equation (3.9) becomes

$$\begin{aligned}(div\mathbb{G})(Y,Z)U = \left[\frac{c}{2} - \frac{1}{n}\left(\frac{a}{n-1} + b + c\right)\right] \times \\ [g(Z,U)dr(Y) - g(Y,U)dr(Z)].\end{aligned} \tag{3.12}$$

Hence, from (3.12), we have the following theorem:

Theorem 3.3. *If* (M^n, g) *be a semi-Riemannian manifold whose Ricci tensor is Codazzi type. Then,* $\mathbb{G}$ *is divergence-free if and only if scalar curvature is constant, provided* $\left[\frac{c}{2} - \frac{1}{n}\left(\frac{a}{n-1} + b + c\right)\right] \neq 0$.

Case (v): A semi-Riemannian manifold (M^n, g) is in subspace $\mathcal{I} \oplus \mathcal{A}$. Then, from (3.7) and (3.9), we get

$$\begin{aligned}(div\mathbb{G})(Y,Z)U =& (a+b)\Big[-2(D_Z Ric)(U,Y) - (D_U Ric)(Z,Y) \\ &+ \frac{2}{n+2}\{dr(Y)g(Z,U) + dr(Z)g(U,Y) + \\ &dr(U)g(Z,Y)\}\Big] + \left[\frac{c}{2} - \frac{1}{n}\left(\frac{a}{n-1} + b + c\right)\right] \times \\ &[g(Z,U)dr(Y) - g(Y,U)dr(Z)].\end{aligned} \tag{3.13}$$

Assuming that $(div\mathbb{G})(Y,Z)U = 0$, then, the contraction of (3.13) with respect to Z and U yields:

$$\left[\left(\frac{a+b}{2}\right) + (n-1)\left\{\frac{c}{2} - \frac{1}{n}\left(\frac{a}{n-1} + b + c\right)\right\}\right] dr(Y) = 0.$$

Hence, we can state the following theorem:

Theorem 3.4. *If* (M^n, g) *be a semi-Riemannian manifold whose Ricci tensor satisfies conformal Killing tensor . The curvature tensor* $\mathbb{G}$ *is divergence-free if and only if either* $\left[\left(\frac{a+b}{2}\right) + (n-1)\left\{\frac{c}{2} - \frac{1}{n}\left(\frac{a}{n-1} + b + c\right)\right\}\right] = 0$ *or scalar curvature is constant.*

Case (vi): Let the curvature tensor $\mathbb{G}$ belong to the class $\mathcal{T} \oplus \mathcal{B}$. Then

$$\begin{aligned}(D_Y Ric)(Z,U) - (D_Z Ric)(Y,U) =&\frac{1}{2(n-1)}[g(Z,U)dr(Y) \\ &- g(Y,U)dr(Z)].\end{aligned} \tag{3.14}$$

Thus, divergence of $\mathbb{G}$ becomes

$$\begin{aligned}(div\mathbb{G})(Y,Z)U = \left[\frac{c}{2} - \frac{1}{n}\left(\frac{a}{n-1} + b + c\right) + \frac{1}{2(n-1)}\right] \times \\ [g(Z,U)dr(Y) - g(Y,U)dr(Z)].\end{aligned} \tag{3.15}$$

For divergence-free $\mathbb{G}$ becomes, we have the following theorem:

Theorem 3.5. *Let (M^n, g) be a semi-Riemannian manifold whose Ricci tensor satisfies (3.14) . The curvature tensor $\mathbb{G}$ is divergence-free if and only if either $[a(n-4) - b(n-2) + c(n-1)(n-2)] = 0$ or scalar curvature is constant.*

4. $(P\mathcal{G}S)_n, n > 2$ satisfies Bianchi's 2nd identity

In this case, we prove that in $(P\mathcal{G}S)_n, \mathcal{G}$ satisfies Bianchi's 2nd identity, that is

$$(D_X\mathcal{G})(Y,Z,U,V) + (D_Y\mathcal{G})(Z,X,U,V) + (D_Z\mathcal{G})(X,Y,U,V) = 0. \tag{4.1}$$

In consequences of (1.4), (1.5) and (4.1), we have

$$\begin{aligned}&(D_X\mathcal{G})(Y,Z,U,V) + (D_Y\mathcal{G})(Z,X,U,V) + (D_Z\mathcal{G})(X,Y,U,V) = \\ &A(U)[\mathcal{G}(Y,Z,X,V) + \mathcal{G}(Z,X,Y,V) + \mathcal{G}(X,Y,Z,V)] + \\ &A(V)[\mathcal{G}(Y,Z,U,X) + \mathcal{G}(Z,X,U,Y) + \mathcal{G}(X,Y,U,Z)].\end{aligned} \tag{4.2}$$

Using (2.7) in (4.2), we get

$$\begin{aligned}&(D_X\mathcal{G})(Y,Z,U,V) + (D_Y\mathcal{G})(Z,X,U,V) + (D_Z\mathcal{G})(X,Y,U,V) = \\ &A(V)[\mathcal{G}(Y,Z,U,X) + \mathcal{G}(Z,X,U,Y) + \mathcal{G}(X,Y,U,Z)].\end{aligned} \tag{4.3}$$

In view of (1.4) and (4.3), we have

$$(D_X\mathcal{G})(Y,Z,U,V) + (D_Y\mathcal{G})(Z,X,U,V) + (D_Z\mathcal{G})(X,Y,U,V) = 0. \tag{4.4}$$

Hence, we can state the following theorem:

Theorem 4.1. *The curvature tensor $\mathcal{G}$, $(P\mathcal{G}S)_n, n > 2$ satisfies Bianchi's 2nd identity.*

5. A $(P\mathcal{G}S)_n, n > 2$ admitting Codazzi type of Ricci tensor

In view of (1.4), we get

$$\begin{aligned}&(D_X\mathcal{G})(Y,Z,U,V) + (D_Y\mathcal{G})(Z,X,U,V) + (D_Z\mathcal{G})(X,Y,U,V) =\\ &b[\{(D_XRic)(Z,U) - (D_ZRic)(X,U)\}g(Y,V) + \{(D_YRic)(X,U)-\\ &(D_XRic)(Y,U)\}g(Z,V) + \{(D_ZRic)(Y,U) - (D_YRic)(Z,U)\}g(X,V)]\\ &+c[\{(D_XRic)(Y,V) - (D_YRic)(X,V)\}g(Z,U) + \{(D_ZRic)(X,V)-\\ &(D_XRic)(Y,V)\}g(Y,U) + \{(D_YRic)(Z,V) - (D_ZRic)(Y,V)\}g(X,U)]\\ &-\frac{1}{n}\left(\frac{a}{n-1}+b+c\right)[\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\}dr(X)+\\ &\{g(X,U)g(Z,V) - g(Z,U)g(X,V)\}dr(Y) + \{g(Y,U)g(X,V)-\\ &g(X,U)g(Y,V)\}dr(Z)].\end{aligned} \tag{5.1}$$

Here, we assume that $(P\mathcal{G}S)_n, n > 2$ admitting Codazzi type of Ricci tensor, then from (5.1), we have

$$\begin{aligned}&(D_X\mathcal{G})(Y,Z,U,V) + (D_Y\mathcal{G})(Z,X,U,V) + (D_Z\mathcal{G})(X,Y,U,V) =\\ &-\frac{1}{n}\left(\frac{a}{n-1}+b+c\right)[\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\}dr(X)+\\ &\{g(X,U)g(Z,V) - g(Z,U)g(X,V)\}dr(Y) + \{g(Y,U)g(X,V)-\\ &g(X,U)g(Y,V)\}dr(Z)].\end{aligned} \tag{5.2}$$

Using (4.4) in (5.2), we obtain

$$\begin{aligned}&\frac{1}{n}\left(\frac{a}{n-1}+b+c\right)[\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\}dr(X)+\\ &\{g(X,U)g(Z,V) - g(Z,U)g(X,V)\}dr(Y) + \{g(Y,U)g(X,V)-\\ &g(X,U)g(Y,V)\}dr(Z)] = 0.\end{aligned} \tag{5.3}$$

Suitable contraction of (5.3) gives

$$[a + (n-1)(b+c)]dr(X) = 0. \tag{5.4}$$

Hence, in view of equation (5.4), we have the following theorem:

Theorem 5.1. *For a $(P\mathcal{G}S)_n, n > 2$ admitting Codazzi type of Ricci tensor, then the scalar curvature is constant, provided $[a + (n-1)(b+c)] \neq 0$.*

From (2.4), we have

$$\mathcal{G}_1(Z,U) = [a + b(n-1) - c]\left[Ric(Z,U) - \frac{r}{n}g(Z,U)\right]. \tag{5.5}$$

In $(P\mathcal{G}S)_n, n > 2$, the $\mathcal{G}$ satisfies (1.5). Then contracting Y and V in (1.5), we get

$$\begin{aligned}(D_X\mathcal{G}_1)(Z,U) =& 2A(X)\mathcal{G}_1(Z,U) + \mathcal{G}(X,Z,U,\rho) + A(Z)\mathcal{G}_1(X,U)\\ &+ A(U)\mathcal{G}_1(Z,X) + \mathcal{G}(\rho,Z,U,X).\end{aligned} \tag{5.6}$$

If $(P\mathcal{G}S)_n$ admitting Codazzi type of Ricci tensor. Then, from (5.6), we get

$$\begin{aligned}&2A(X)\mathcal{G}_1(Z,U) + \mathcal{G}(X,Z,U,\rho) + A(Z)\mathcal{G}_1(X,U) + A(U)\mathcal{G}_1(Z,X)\\ &+ \mathcal{G}(\rho,Z,U,X) = 2A(Z)\mathcal{G}_1(X,U) + \mathcal{G}(Z,X,U,\rho) +\\ &A(X)\mathcal{G}_1(Z,U) + A(U)\mathcal{G}_1(X,Z) + \mathcal{G}(\rho,X,U,Z) = 0.\end{aligned} \tag{5.7}$$

Solving (5.7), we get

$$\begin{aligned}&A(X)\mathcal{G}_1(Z,U) - A(Z)\mathcal{G}_1(X,U) +\\ &2\mathcal{G}(X,Z,U,\rho) + \mathcal{G}(\rho,Z,U,X) - \mathcal{G}(\rho,X,U,Z) = 0.\end{aligned} \tag{5.8}$$

Contracting (5.8) with respect to Z and U, we get

$$[2a - b(n+2) + c(3n-2)]\left[Ric(X,\rho) - \frac{r}{n}g(X,\rho)\right] = 0. \tag{5.9}$$

Hence from (5.9), we have the following theorem:

Theorem 5.2. *In $(P\mathcal{G}S)_n, n > 2$ admitting Codazzi type of Ricci tensor, then $\frac{r}{n}$ is an eigen value of the Ricci tensor Ric corresponding to the eigen vector ρ, provided $[2a - b(n+2) + c(3n-2)] \neq 0$.*

6. Einstein $(P\mathcal{G}S)_n, n > 2$

This section deals with an Einstein $(P\mathcal{G}S)_n, n > 2$. Then, the Ricci tensor satisfies

$$Ric(U,V) = \frac{r}{n}g(U,V), \tag{6.1}$$

from which it follows that

$$dr(U) = 0 \quad \text{and} \quad (D_Z Ric)(U,V) = 0, \tag{6.2}$$

for all vector fields U, V and Z.
Using (6.2) in (1.4), we get

$$(D_X\mathcal{G})(Y,Z,U,V) = a(D_XR)(Y,Z,U,V). \tag{6.3}$$

In view of (1.5) and (6.3), we get

$$\begin{aligned} a(D_XR)(Y,Z,U,V) =& 2A(X)\mathcal{G}(Y,Z,U,V) + A(Y)\mathcal{G}(X,Z,U,V)+ \\ & A(Z)\mathcal{G}(Y,X,U,V) + A(U)\mathcal{G}(Y,Z,X,V) \\ & + A(V)\mathcal{G}(Y,Z,U,X), \end{aligned} \tag{6.4}$$

From (1.4), (6.1) and (6.4), we get

$$\begin{aligned} a(D_XR)(Y,Z,U,V) =& 2A(X)\Big[aR(Y,Z,U,V) - \frac{ra}{n(n-1)}\times \\ & \{g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\}\Big]+ \\ & A(Y)\Big[aR(X,Z,U,V) - \frac{ra}{n(n-1)}\times \\ & \{g(Z,U)g(X,V) - g(X,U)g(Z,V)\}\Big]+ \\ & A(Z)\Big[aR(Y,X,U,V) - \frac{ra}{n(n-1)}\times \\ & \{g(X,U)g(Y,V) - g(Y,U)g(X,V)\}\Big]+ \\ & A(U)\Big[aR(Y,Z,X,V) - \frac{ra}{n(n-1)}\times \\ & \{g(Z,X)g(Y,V) - g(Y,X)g(Z,V)\}\Big] \\ & + A(V)\Big[aR(Y,Z,U,X) - \frac{ra}{n(n-1)}\times \\ & \{g(Z,U)g(Y,X) - g(Y,U)g(Z,X)\}\Big]. \end{aligned} \tag{6.5}$$

If the Einstein $(P\mathcal{G}S)_n, n > 2$ is $(PS)_n$ with the same associated form A, then from (6.5), we get

$$\begin{aligned} & r\big[2A(X)\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\}+ \\ & A(Y)\{g(Z,U)g(X,V) - g(X,U)g(Z,V)\}+ \\ & A(Z)\{g(X,U)g(Y,V) - g(Y,U)g(X,V)\}+ \\ & A(U)\{g(Z,X)g(Y,V) - g(Y,X)g(Z,V)\}+ \\ & A(V)\{g(Z,U)g(Y,X) - g(Y,U)g(Z,X)\}\big] = 0. \end{aligned} \tag{6.6}$$

Contracting (6.6) with respect to Y and V, we get

$$\begin{aligned}&r\big[2(n-1)A(X)g(Z,U)+\{A(X)g(Z,U)-A(Z)g(X,U)\}+\\&(n-1)\{A(Z)g(X,U)-A(U)g(Z,X)\}+\\&\{A(X)g(Z,U)-A(U)g(Z,X)\}\big]=0.\end{aligned}\tag{6.7}$$

Again, contracting (6.7) with respect to Z and U, we get

$$(n-1)(n+2)rA(X)=0,\ \textit{provided } a\neq 0.$$

Then, it follows that $A(X)=0$, which is contradiction to our fact. Therefore, only possibility is $r=0$.
Thus, we conclude the following theorem:

Theorem 6.1. *If an Einstein $(P\mathcal{G}S)_n, n>2$ is $(PS)_n$, then the scalar curvature of the manifold vanishes, $a\neq 0$.*

Again, in an Einstein $(P\mathcal{G}S)_n, n>2$, if $r=0$, then from (6.5), we get

$$\begin{aligned}(D_XR)(Y,Z,U,V)=&2A(X)R(Y,Z,U,V)+A(Y)R(X,Z,U,V)+\\&A(Z)R(Y,X,U,V)+A(U)R(Y,Z,X,V)\\&+A(V)R(Y,Z,U,X).\end{aligned}$$

Hence, we can state the following theorem:

Theorem 6.2. *If in an Einstein $(P\mathcal{G}S)_n, n>2$ with $a\neq 0$, the scalar curvature vanishes, then the manifold is a $(PS)_n$.*

7. Spacetime with vanishing generalized quasi-conformal curvature tensor $\mathcal{G}$

Let (M^4,g) be the spacetime of $\widetilde{GR}$, then from (1.4), we get for flat manifold

$$\begin{aligned}&aR(X,Y,U,V)+b[Ric(Y,U)g(X,V)-Ric(X,U)g(Y,V)]\\&+c[g(Y,U)Ric(X,V)-g(X,U)Ric(Y,V)]-\\&\frac{r}{4}\left[\frac{a}{3}+b+c\right][g(Y,U)g(X,V)-g(X,U)g(Y,V)]=0.\end{aligned}\tag{7.1}$$

Contracting (7.1) with respect to X and V, we get

$$[a+3b-c]\left[Ric(Y,U)-\frac{r}{4}g(Y,U)\right]=0.\tag{7.2}$$

If $[a+3b-c]\neq 0$, then

$$Ric(Y,U)=\frac{r}{4}g(Y,U).\tag{7.3}$$

Thus, we have the following theorem:

Theorem 7.1. *A generalized quasi-conformally flat spacetime is an Einstein spacetime, provided* $(a+3b-c) \neq 0$.

From (7.2) and (7.3), we get

$$R(X,Y,U,V) = \frac{r}{12}[g(Y,U)g(X,V) - g(X,U)g(Y,V)] \ \text{provided } a \neq 0. \tag{7.4}$$

It follows that the manifold (M^n, g) is constant curvature.
Thus, we can state the following Corollary:

Corollary 7.2. *A generalized quasi-conformally flat spacetime is a spacetime of constant curvature, provided* $a \neq 0$.

Let us consider a spacetime satisfying the Einstein's field equation with cosmological constant λ

$$Ric(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = k\mathcal{T}(X,Y), \tag{7.5}$$

where symbols have their usual meanings.

From (7.3) and (7.5), we get

$$\mathcal{T}(X,Y) = \frac{1}{k}\left(\lambda - \frac{r}{4}\right) g(X,Y). \tag{7.6}$$

Taking the covariant derivative of (7.6), we get

$$(D_Z\mathcal{T})(X,Y) = -\frac{1}{4k}dr(Z)g(X,Y). \tag{7.7}$$

According to our assumption, we have

$$dr(X) = 0,$$

for all X. Hence, from above equation (7.7) gives $(D_Z\mathcal{T})(X,Y) = 0$.

Thus, we have the following theorem:

Theorem 7.3. *In a generalized quasi-conformally flat spacetime satisfying EFE with* λ*, then EMT is covariant constant, provided* $(a+3b-c) \neq 0$.

The curvature collineation studied by Katzin et al. [15], in context of the related particle and field conservation laws that may be admitted in the standard form of $\widetilde{GR}$.

Geometrical symmetries of a spacetime is

$$L_\xi \overline{B} - 2\psi \overline{B} = 0, \tag{7.8}$$

where $\overline{B}$ represents a geometrical physical quantity, L_ξ denotes the Lie derivative with respect to vector field ξ and ψ is a scalar.

One of the most simplest and widely used example is the metric inheritance symmetry for $\overline{B} = g(X, Y)$ in (7.8) and ξ is the Killing vector field if $\psi = 0$. That is

$$(L_\xi g)(X, Y) = 2\psi g(X, Y). \tag{7.9}$$

A spacetime of the manifold (M^4, g) is said to be admit a symmetry called a curvature collineation, provided there exist a vector field ξ such that [10]

$$(L_\xi \mathbb{R})(X, Y)Z = 0, \tag{7.10}$$

where $\mathbb{R}$ is the Riemannian curvature tensor of the type (1,3).

Now, we investigate the role of such symmetry inheritance for the spacetime admitting generalized quasi-conformal curvature tensor $\mathbb{G}$ with a Killing vector ξ as a curvature collineation. Then, we have

$$(L_\xi g)(X, Y) = 0. \tag{7.11}$$

If (M^4, g) admits a curvature collineation and then (7.10) gives

$$(L_\xi Ric)(X, Y) = 0. \tag{7.12}$$

Taking the Lie derivative of (1.3) and using (7.10), (7.11), (7.12), we get

$$(L_\xi \mathbb{G})(X, Y)U = 0. \tag{7.13}$$

Hence, we can state the following theorem:

Theorem 7.4. *If a spacetime (M^4, g) admitting the generalized quasi-conformal curvature tensor $\mathbb{G}$ with ξ is a Killing vector field is the curvature collineation, then the Lie derivative of $\mathbb{G}$ vanishes along the vector field ξ.*

Next, the symmetry of the EMT $\mathcal{T}$ is the matter collineation defined by

$$(L_\xi \mathcal{T})(X, Y) = 0, \tag{7.14}$$

where ξ is the vector field generating the symmetry and L_ξ is the Lie derivative operator along the vector field on the spacetime with vanishing $\mathbb{G}$, then

$$(L_\xi g)(X, Y) = 0. \tag{7.15}$$

Taking the Lie derivative of the equation (7.5), we get

$$(L_\xi \mathcal{T})(X, Y) = \frac{1}{k}\left(\lambda - \frac{r}{4}\right)(L_\xi g)(X, Y). \tag{7.16}$$

From (7.15) and (7.16), we get

$$(L_\xi \mathcal{T})(X, Y) = 0, \tag{7.17}$$

which implies that the spacetime admits matter collineation. Conversely, if $(L_\xi \mathcal{T})(X, Y) = 0$, then (7.16) becomes

$$(L_\xi g)(X, Y) = 0,$$

Hence, we can state the following theorem:

Theorem 7.5. *If a spacetime obeying EFE has vanishing $\mathbb{G}$, then the spacetime admits matter collineation with respect to a vector field ξ if and only if ξ is Killing vector field, provided scalar curvature r is non-zero.*

If ξ is a conformal Killing vector field, then

$$(L_\xi g)(X, Y) = 2\alpha g(X, Y), \tag{7.18}$$

where α is a scalar.

Using (7.18) in (7.16), we get

$$\frac{k(L_\xi \mathcal{T})(X, Y)}{\left(\lambda - \frac{r}{4}\right)} = 2\alpha g(X, Y), \tag{7.19}$$

From (7.5) and (7.19), we get

$$(L_\xi \mathcal{T})(X, Y) = 2\alpha \mathcal{T}(X, Y). \tag{7.20}$$

Therefore, from (7.20), we can say that the EMT has Lie inheritance property along ξ.

Conversely, if (7.20) holds, then it follows that (7.18), that is ξ is a conformal Killing vector field.

Thus, we can state the following theorem:

Theorem 7.6. *If a spacetime obeying EFE has vanishing $\mathbb{G}$, then the vector field ξ is a conformal Killing vector field if and only if the EMT has the Lie inheritance property along ξ.*

8. Dust fluid spacetime with vanishing generalized quasi-conformal curvature tensor

In a dust or pressureless fluid spacetime, the EMT is of the form [29]

$$\mathcal{T}(X,Y) = \delta A(X)A(Y). \tag{8.1}$$

where δ is the energy density of the dust-like matter and A is non-zero 1-form such $g(X,\xi) = A(X)$, for all X, ξ be the velocity vector field of the flow i.e. $g(\xi,\xi) = -1$.

Using (7.6) in (8.1), we get

$$\left(\lambda - \frac{r}{4}\right) g(X,Y) = k\delta A(X)A(Y). \tag{8.2}$$

Taking frame field and contracting over X and Y, we get

$$\lambda = \frac{r}{4} - \frac{k\delta}{4}. \tag{8.3}$$

Again, replacing X and Y be ξ in (8.2), we get

$$\lambda = \delta k + \frac{r}{4}. \tag{8.4}$$

From (8.3) and (8.4), we get

$$\delta = 0, \quad k \neq 0. \tag{8.5}$$

Hence, in view of (8.1) and (8.5), we get

$$\mathcal{T}(X,Y) = 0. \tag{8.6}$$

Hence, we can state the following theorem:

Theorem 8.1. *A generalized quasi-conformally flat dust fluid spacetime satisfying EFE with cosmological constant is vacuum.*

9. Example

Let us consider a Lorentzian manifold (M^4, g) endowed with the Lorentzian metric g is

$$ds^2 = g_{ij}dx^i dx^j = sinx^2(dx^1)^2 + sinx^3(dx^2)^2 - (dx^3)^2 + (dx^4)^2 \tag{9.1}$$

where $i, j = 1, 2, 3, 4$ and $sinx^2,\ sinx^3$ are non zero.
Then the only non-vanishing components of the Christoffel symbols are

$$\Gamma^2_{11} = -\frac{cosx^2}{2sinx^3}, \quad \Gamma^3_{22} = \frac{1}{2}cosx^3, \quad \Gamma^1_{12} = \frac{1}{2}cotx^2 \quad \Gamma^2_{23} = \frac{1}{2}cotx^3, \tag{9.2}$$

The curvature tensor of type (1,3) is defined by

$$\mathbb{R}^a_{ijk} = -\frac{\partial}{\partial x^k}\Gamma^a_{ij} + \frac{\partial}{\partial x^j}\Gamma^a_{ik} + \Gamma^a_{bj}\Gamma^b_{ik} - \Gamma^a_{bk}\Gamma^b_{ij}. \tag{9.3}$$

In view of (9.2) and (9.3), we get

$$\mathbb{R}^1_{221} = -\frac{1}{2} - \frac{1}{4}cot^2x^2, \quad \mathbb{R}^3_{112} = \frac{1}{4}cotx^3cosx^2. \tag{9.4}$$

The curvature tensor of type (0,4) is defined by

$$R_{hijk} = g_{ha}\mathbb{R}^a_{ijk}. \tag{9.5}$$

In view of (9.4) and (9.5), we get

$$R_{3112} = -\frac{1}{4}cotx^3cosx^2, \quad R_{1221} = -\frac{1}{4}cosx^2 - \frac{1}{2}sinx^2, \tag{9.6}$$

The non-vanishing components of the Ricci tensor are

$$R_{11} = -\frac{cosx^2 + 2sinx^2}{4sinx^3}, \quad R_{23} = -\frac{cotx^2cotx^3}{4}, \tag{9.7}$$

and scalar curvature is

$$R = -\frac{cotx^2 + 2}{4sinx^3}.$$

We shall now show that this M^4 is a pseudo generalized quasi-conformal symmetric spacetime i.e., it satisfies the defining relation (1.5).
Now, only the non vanishing component for $\mathcal{G}$ and its covariant derivatives are given by

$$\begin{aligned} &\mathcal{G}_{3112} = -\left(\frac{a+c}{4}\right)cotx^3cosx^2, \quad \mathcal{G}_{3112,2} = -\left(\frac{a+c}{4}\right)cosecx^2cotx^3, \\ &\mathcal{G}_{3112,3} = \left(\frac{a+c}{4}\right)[cot^2x^3 + cosec^2x^3]cosx^2, \end{aligned} \tag{9.8}$$

Let us choose the associated 1-form as

$$A_i = \begin{cases} 0, & i = 1, 4 \\ -\frac{1}{3} cosecx^2 secx^2, & i = 2 \\ -\left(\frac{\frac{1}{2} cot^2 x^3 + cosec^2 x^3}{3 cot x^3}\right), & i = 3 \end{cases} \tag{9.9}$$

at any point $x \in (M^4, g)$. In (M^4, g), after using (9.8) and (9.9) in R.H.S. of (1.5), we obtain

$$\begin{aligned} & 2A_2\mathcal{G}_{3112} + A_3\mathcal{G}_{2112} + A_1\mathcal{G}_{3212} + A_1\mathcal{G}_{3122} + A_2\mathcal{G}_{3112} = \\ & 2\left[-\frac{1}{3} cosecx^2 secx^2\right] \times \left[-\frac{a+c}{4} cotx^3 cosx^2\right] + 0 + 0 \\ & + 0 + \left[-\frac{1}{3} cosecx^2 secx^2\right] \times \left[-\frac{a+c}{4} cotx^3 cosx^2\right] \\ & = -\left(\frac{a+c}{4}\right) cosecx^2 cotx^3, \end{aligned} \tag{9.10}$$

In view of (9.8) and (9.10), we get

$$\mathcal{G}_{3112,2} = 2A_2\mathcal{G}_{3112} + A_3\mathcal{G}_{2112} + A_1\mathcal{G}_{3212} + A_1\mathcal{G}_{3122} + A_2\mathcal{G}_{3112}. \tag{9.11}$$

Similarly, we can prove that

$$\mathcal{G}_{3112,3} = 2A_3\mathcal{G}_{3112} + A_3\mathcal{G}_{3112} + A_1\mathcal{G}_{3312} + A_1\mathcal{G}_{3132} + A_2\mathcal{G}_{3113}. \tag{9.12}$$

Hence, It is verified that the equations (9.11) and (9.12) are true. Thus, the manifold under consideration is a pseudo generalized quasi-conformal symmetric spacetime, that is, $(P\mathcal{G}S)_4$.
Hence, we can state the following:

Theorem 9.1. *Let* (M^4, g) *be Riemannian manifold endowed with the metric given by*

$$ds^2 = g_{ij}dx^i dx^j = sinx^2(dx^1)^2 + sinx^3(dx^2)^2 - (dx^3)^2 + (dx^4)^2,$$

where $i, j = 1, 2, 3, 4$ *and* $sinx^2$, $sinx^3$ *are non zero. Then* (M^4, g) *is a* $(P\mathcal{G}S)_4$.

10. Conclusion

In this paper, we have studied the generalized quasi-conformal curvature tensor $\mathcal{G}$, which unifies several classical curvature tensors. Basic properties of $\mathcal{G}$ were obtained, and its geometric significance was analyzed.

Using Gray's decomposition, we derived necessary and sufficient conditions for $\mathbb{G}$ to be divergence-free in different invariant subspaces. We also investigated pseudo generalized quasi-conformally symmetric manifolds $(P\mathcal{G}S)_n$ and proved that such manifolds satisfy the second Bianchi identity. For manifolds admitting a Codazzi type Ricci tensor, important results regarding the constancy of scalar curvature and eigenvalue structure of the Ricci tensor were established.

Further, in the case of Einstein $(P\mathcal{G}S)_n$ manifolds, it is shown that the scalar curvature vanishes under suitable conditions, and conversely, such manifolds reduce to pseudo-symmetric spaces. In the context of four-dimensional spacetime, we proved that generalized quasi-conformally flat spacetimes are Einstein and, in particular cases, of constant curvature. Additional results related to the energy-momentum tensor, curvature collineations, and matter collineations were also obtained.

Finally, a non-trivial example of a Lorentzian $(P\mathcal{G}S)_4$ manifold was constructed to illustrate the theoretical findings.

Acknowledgement

The authors would like to express their deep thanks to the referees for his careful reading and many valuable suggestions towards the improvement of this paper.

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